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Bireflectionality of the Weak Orthogonal and the Weak Symplectic Groups

E. W. ELLERS*

Department of Mathematics, University of Toronto, Toronto, Ontario M5S 1A1, Canada

AND

R. FRANK AND W. NOLTE

Fachbereich Mathematik, AG 2 Geometrie und Algebra, Technische Hochschule Darmstadt, Schlossgartenstrasse 7, D-6100 Darmstadt, West Germany

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1. INTRODUCTION

Let V be a finite-dimensional vector space, Q a quadratic form and $f_Q = f$ the bilinear form associated with Q. We assume (V, f_Q) is regular. Then the orthogonal group $O(V)$ is bireflectional, i.e., every isometry in $O(V)$ is a product of two involutory isometries in $O(V)$. This has been shown in [8] if the field of scalars K has characteristic distinct from 2 and in [4] and [5] if $\operatorname{char} K = 2$. The latter papers also establish the bireflectionality for the symplectic group $\operatorname{Sp}(V)$, again under the assumption that (V, f) is regular and $\operatorname{char} K = 2$. For $\operatorname{char} K \neq 2$ the symplectic group is not bireflectional (see [3]).

We shall extend the results just mentioned. We shall drop the assumptions that V is finite dimensional and that (V, f) is regular, i.e., the vector space V may be infinite dimensional and the radical of V may be distinct from zero.

We shall use the notation and the concepts in [2]. For every $\pi \in \operatorname{Hom}(V, V)$ we define $F(\pi) = \{v \in V; v\pi = v\}$ and $B(\pi) = \{v\pi - v; v \in V\}$. The spaces $F(\pi)$ and $B(\pi)$ are called fix and path of π, respectively.

The groups $O^*(V) = \{\pi \in O(V);\ \operatorname{rad} V \subset F(\pi) \text{ and } \dim B(\pi) < \infty\}$ and $\operatorname{Sp}^*(V) = \{\pi \in \operatorname{Sp}(V);\ \operatorname{rad} V \subset F(\pi) \text{ and } \dim B(\pi) < \infty\}$ are called the weak orthogonal and the weak symplectic group, respectively.

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In Section 2 we shall assume that the dimension of the vector space V is finite. We shall show that under this assumption the weak orthogonal group and for char $K = 2$ also the weak symplectic group are bireflectional. Since for char $K \neq 2$ the symplectic group is not bireflectional, we shall always assume that char $K = 2$ when we deal with the weak symplectic group $\mathrm{Sp}^*(V)$.

In Section 3 we investigate the situation in infinite-dimensional vector spaces. First we see in Lemma 3 that for char $K \neq 2$ every involution in $O^*(V)$ and therefore every product of two involutions is a product of reflections. Since there are elements in $O^*(V)$ which are not products of reflections (see [2, p. 116]) we see that $O^*(V)$ is not bireflectional. This suggests to define $O^{**}(V)$ as the subgroup of $O(V)$ whose elements are products of reflections and to give an analogous definition for $\mathrm{Sp}^{**}(V)$. We shall show in Theorem 4 that $O^{**}(V)$ and $\mathrm{Sp}^{**}(V)$ are bireflectional if $K \neq GF(2)$.

Even if $K = GF(2)$, we see in Theorem 4 that the elements in $O^{**}(V)$ and $\mathrm{Sp}^{**}(V)$ are products of two involutions in $O^*(V)$ and $\mathrm{Sp}^*(V)$, respectively.

Finally we need a few formulas whose proof is not difficult (see, e.g., [7]): For $\pi_1, \pi_2 \in \mathrm{Hom}(V, V)$ and $\pi \in GL(V)$ we get

$$B(\pi_1 \pi_2) \subset B(\pi_1) + B(\pi_2), \tag{1}$$

$$B(\pi^{-1}) = B(\pi), \tag{2}$$

$$B(\pi^m) \subset B(\pi) \qquad \text{for all integers } m. \tag{3}$$

2. The Groups $O^*(V)$ and $\mathrm{Sp}^*(V)$ for a Finite-Dimensional Vector Space V

In this section we assume that V is a finite-dimensional vector space. For the proof of Theorem 1 we embed V into a regular vector space $\bar{V}$. Then we apply results of [4] and [8]. Before we can do that it is necessary to learn more about the involutory factors ρ_1 and ρ_2 of π which were constructed in [4] and [8]. We need the following result:

Assume V is regular. Then every isometry π is a product of two involutory isometries ρ_1 and ρ_2 such that

$$B(\rho_i) \subset B(\pi), \qquad i = 1, 2. \tag{4}$$

We shall see that (4) is true for the involutory isometries ρ_1 and ρ_2 in [4] (see Lemma 1 and the proof of Lemma 4) and [8] (see the proofs of Lemmas 2 and 5). The vector space V is an orthogonal sum of regular π-cyclic and regular π-bicyclic subspaces. Since a π-bicyclic subspace is a direct sum of two (nonregular) π-cyclic subspaces, we get that V is a direct sum of π-cyclic subspaces W_k: $V = \sum_{k=1}^{q} W_k$ for some integer q. Let w_k be a π-cyclic generator of W_k and assume $\dim W_k = n_k$. Then $\{w_k \pi^i; k = 1,..., q; i = 0,..., n_{k-1}\}$ is a basis for V.

First we shall deal with [4]. The restriction of ρ_1 to the π-cyclic subspace W_k is one of the two involutory mappings given in [4], (7), which we now call σ_1 and σ_2. We give their definitions (putting $w_k = w$ and $n_k = n$):

$$w\pi^i\sigma_1 = w\pi^{n-i} \qquad \text{and} \qquad w\pi^i\sigma_2 = w\pi^{n+1-i} \qquad \text{for all integers } i. \tag{5}$$

Then $w(\pi^i\sigma_1 - \pi^i) = w(\pi^{n-i} - \pi^i) = w\pi^i(\pi^{n-2i} - 1) \in B(\pi^{n-2i}) \subset B(\pi)$ by (3) and consequently $B(\sigma_1) \subset B(\pi)$. A similar calculation shows $B(\sigma_2) \subset B(\pi)$. Therefore $B(\rho_1) \subset B(\pi)$.

Second we consider [8] (i.e., char $K \neq 2$). In [8] the path of ρ_1 is always generated by vectors of the form $u(\pi^i - \pi^{-i})$ for some $u \in V$. Now $u(\pi^i - \pi^{-i}) = u\pi^{-i}(\pi^{2i} - 1) \in B(\pi^{2i}) \subset B(\pi)$. Thus again $B(\rho_1) \subset B(\pi)$.

Using (2) we get $B(\rho_2) \subset B(\rho_1) + B(\pi) \subset B(\pi)$ and therefore (4) holds.

Now we are ready to state and prove our first result. We allow V to be nonregular.

THEOREM 1. *Assume V is a finite-dimensional vector space. Then the weak orthogonal group $O^*(V)$ is bireflectional and if* char $K = 2$, *also the weak symplectic group* $\mathrm{Sp}^*(V)$ *is bireflectional.*

Proof. Let $V = A \oplus \operatorname{rad} V$ (then A is regular) and $k = \dim \operatorname{rad} V$. Then V can be embedded into a regular vector space $\bar{V}$ such that $\dim \bar{V} = \dim V + k$. By Witt's theorem (see [1, p. 71]) every isometry π of V can be extended to an isometry $\bar{\pi}$ of $\bar{V}$. In $\bar{V}$ we have $k + \dim(\operatorname{rad} V)^\perp = \dim \operatorname{rad} V + \dim(\operatorname{rad} V)^\perp = \dim \bar{V} = \dim V + k$. Hence $\dim(\operatorname{rad} V)^\perp = \dim V$. This implies $V = (\operatorname{rad} V)^\perp$ since $V \subset (\operatorname{rad} V)^\perp$. Clearly $\operatorname{rad} V \subset F(\pi) \subset F(\bar{\pi})$ and therefore $V = (\operatorname{rad} V)^\perp \supset F(\bar{\pi})^\perp = B(\bar{\pi})$.

By [4] and [8] there are involutions $\bar{\rho}_1, \bar{\rho}_2 \in O(\bar{V})$ and $\bar{\rho}_1, \bar{\rho}_2 \in \mathrm{Sp}(\bar{V})$, respectively, such that $\bar{\pi} = \bar{\rho}_1\bar{\rho}_2$. By (4) we have $B(\bar{\rho}_i) \subset B(\bar{\pi}) \subset V$; thus V is invariant under $\bar{\rho}_i$ for $i = 1, 2$. Let $\rho_i = {}_V|\bar{\rho}_i$, then $\pi = \rho_1\rho_2$.

Since $\bar{V}$ is regular and since $B(\bar{\rho}_i) \subset V$ we get $F(\bar{\rho}_i) = B(\bar{\rho}_i)^\perp \supset V^\perp$. Finally $\operatorname{rad} V = V \cap V^\perp \subset V \cap F(\bar{\rho}_i) = F(\rho_i)$.

COROLLARY 2. *Assume V is a finite-dimensional vector space. Then $O^*(V) \cup (-1) \cdot O^*(V)$ is a bireflectional group.*

Proof. Clearly $O^*(V) \cup (-1) \cdot O^*(V)$ is a group. If $\pi \in (-1) \cdot O^*(V)$, then $-\pi \in O^*(V)$. Thus $-\pi = \rho_1\rho_2$, where $\rho_i \in O^*(V)$ are involutions. Therefore $\pi = (-\rho_1)\rho_2$, $(-\rho_1) \in (-1) \cdot O^*(V)$ and $-\rho_1$ is an involution.

3. The Subgroups Generated by Reflections or Symplectic Transvections

In this section the dimension of V is arbitrary, it may be infinite.

First we observe that $O^*(V)$ is not bireflectional if $\dim V/R$ is infinite, $R \neq \{O\}$, and char $K \neq 2$. This is a consequence of the following Lemma 3 and [2, p. 116].

Lemma 3. *Assume* char $K \neq 2$. *If $\rho \in O^*(V)$ is an involution, then ρ is a product of reflections.*

Proof. Since ρ is an involution and since char $K \neq 2$, we have $B(\rho) \cap F(\rho) = \{0\}$. Hence $B(\rho) \cap \operatorname{rad} V = \{0\}$ since $\operatorname{rad} V \subset F(\rho)$. Therefore $B(\rho)^\perp = F(\rho)$ by [2, p. 105]. Thus $\operatorname{rad} B(\rho) = B(\rho)^\perp \cap B(\rho) = F(\rho) \cap B(\rho) = \{0\}$. Hence $B(\rho)$ is regular. Consequently there is an orthogonal basis $\{a_1, \ldots, a_m\}$ for $B(\rho)$ and $\rho = \sigma_1 \cdots \sigma_m$, where σ_i is a reflection such that $B(\sigma_i) = Ka_i$ for $i = 1, \ldots, m$.

Lemma 3 is not true for char $K = 2$. A counterexample can easily be constructed using the example given in [2, p. 116].

Our aim is to show that suitable subgroups of $O^*(V)$ and $\mathrm{Sp}^*(V)$ are bireflectional, namely, the groups $O^{**}(V)$ and $\mathrm{Sp}^{**}(V)$: $O^{**}(V)$ consists of all elements in $O(V)$ that are products of reflections; $\mathrm{Sp}^{**}(V)$ consists of all elements in $\mathrm{Sp}(V)$ that are products of symplectic transvections τ_i such that $B(\tau_i) \cap \operatorname{rad} V = \{0\}$ and char $K = 2$.

Clearly, $O^{**}(V) \subset O^*(V)$ and $\mathrm{Sp}^{**}(V) \subset \mathrm{Sp}^*(V)$.

Theorem 4. *For $\pi \in O^{**}(V)$ and $\pi \in \mathrm{Sp}^{**}(V)$ there are involutory isometries $\rho_i \in O^*(V)$ and $\rho_i \in \mathrm{Sp}^*(V)$, respectively, $i = 1, 2$, such that $\pi = \rho_1 \rho_2$. If $K \neq GF(2)$, then $O^{**}(V)$ and $\mathrm{Sp}^{**}(V)$ are bireflectional.*

Proof. By [2, p. 105], there is some subspace $D \subset V$ such that $B(\pi)^\perp = F(\pi) \oplus D$ and $D(\pi - 1) = B(\pi) \cap \operatorname{rad} V$. If there is some $d_1 \in D \backslash \{0\}$, then there is some hyperplane H_1 such that $F(\pi) \subset H_1$ and $d_1 \notin H_1$. Put $D_1 = D \cap H_1$. Then $\dim D_1 = \dim D - 1$ and $D_1 \cap F(\pi) = \{0\}$. Continuing this process we get hyperplanes $H_1, \ldots, H_m$ where $m = \dim D$ such that $D \cap H_1 \cap \cdots \cap H_m = \{0\}$ and $F(\pi) \subset H_1 \cap \cdots \cap H_m$. By [2, Lemma 18], there are $t_i \in V$ such that $H_i = t_i^\perp$ for $i = 1, \ldots, m$. Put $T = \langle t_1, \ldots, t_m \rangle$. Then $F(\pi) \subset T^\perp$ and $D \cap T^\perp = \{0\}$.

Suppose C is a complement of $F(\pi)$, i.e., $V = F(\pi) \oplus C$. Let $X = B(\pi) + T + C$. Then X is π-invariant, finite dimensional, and $X^\perp = B(\pi)^\perp \cap T^\perp \cap C^\perp = (D + F(\pi)) \cap T^\perp \cap C^\perp = F(\pi) \cap C^\perp \subset F(\pi)$.

There is some finite-dimensional subspace W of V such that $X \subset W$ and $\operatorname{rad} W \subset \operatorname{rad} V$. In order to see that, let $X = Y \oplus \operatorname{rad} X$ and if there is some $x \in \operatorname{rad} X \backslash \operatorname{rad} V$, then $V = Y \oplus Y^\perp$ and there is some $z \in Y^\perp$ such that $f(x, z) \neq 0$ and therefore $\dim \operatorname{rad}(X + Kz) \leqslant \dim \operatorname{rad} X - 1$. If $\operatorname{rad}(X + Kz) \subset \operatorname{rad} V$, put $X + Kz = W$. Otherwise repeat the construction.

Now $V = W + W^{\perp}$ and $W^{\perp} \subset X^{\perp} \subset F(\pi)$.

Since $\operatorname{rad} W \subset \operatorname{rad} V$, we have ${}_{\operatorname{rad} W}|\pi = 1$. Using Theorem 1 we get ${}_{W}|\pi = \tilde{\rho}_1 \tilde{\rho}_2$, where for $i = 1, 2$ $\tilde{\rho}_i$ are involutory isometries of W and ${}_{\operatorname{rad} W}|\tilde{\rho}_i = 1$. Therefore we can define the desired involutory isometries ρ_i by ${}_{W}|\rho_i = \tilde{\rho}_i$ and ${}_{W^{\perp}}|\rho_i = 1$. Clearly, $B(\rho_i) = B(\tilde{\rho}_i)$, $\dim B(\rho_i) < \infty$, $V^{\perp} \subset W^{\perp} \subset F(\rho_i)$, so $\rho_i \in O^*(V)$, or $\mathrm{Sp}^*(V)$, and $\rho_1 \rho_2 = \pi$.

Now we assume $K \neq GF(2)$. This together with the fact that $\tilde{\rho}_i \in O^*(W)$ or $\mathrm{Sp}^*(W)$ for the finite-dimensional space W implies by [2, Theorem 33 and 24], respectively, that $\tilde{\rho}_i$ is a product of reflections or regular symplectic transvections, respectively. These reflections and transvections can be extended in a natural way to reflections and transvections of V, respectively. Then ρ_i is a product of these reflections or transvections, respectively. This proves our last contention.

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